First-Order Logic

Part Two

Recap from Last Time

What is First-Order Logic?

- ***First-order logic*** is a logical system for reasoning about properties of objects.
- Augments the logical connectives from propositional logic with
  - ***predicates*** that describe properties of objects,
  - ***functions*** that map objects to one another, and
  - ***quantifiers*** that allow us to reason about many objects at once.

Some spider is radioactive.

$$\boxed{\exists s.}\ (Spider(s) \land Radioactive(s))$$

∃ is the ***existential quantifier*** and says "for some choice of s, the following is true."

“Some A is a B”

translates as

$$\exists x.\ (A(x) \land B(x))$$

Useful Intuition:

Existentially-quantified statements are false unless there's a positive example.

$$\exists x.\ (A(x) \land B(x))$$

If *x* is an example, it *must* have property *A* on top of property *B*.

"For any natural number n,
n is even if and only if n^2 is even"

$$\forall n.\ (n \in \mathbb{N} \to (Even(n) \leftrightarrow Even(n^2)))$$

∀ is the ***universal quantifier*** and says "for any choice of n, the following is true."

“All *A*'s are *B*'s”

translates as

$$\forall x.\ (A(x) \to B(x))$$

Useful Intuition:

Universally-quantified statements are true unless there's a counterexample.

$$\forall x.\ (A(x) \to B(x))$$

If *x* is a counterexample, it *must* have property A but not have property *B*.

New Stuff!

The Aristotelian Forms

"All *A*s are *B*s"

$$\forall x.\ (A(x) \to B(x))$$

"Some *A*s are *B*s"

$$\exists x.\ (A(x) \land B(x))$$

"No *A*s are *B*s"

$$\forall x.\ (A(x) \to \neg B(x))$$

"Some *A*s aren't *B*s"

$$\exists x.\ (A(x) \land \neg B(x))$$

It is worth committing these patterns to memory. We'll be using them throughout the day and they form the backbone of many first-order logic translations.

The Art of Translation

Using the predicates

- $Person(p)$, which states that p is a person, and
- $Loves(x, y)$, which states that x loves y,

write a sentence in first-order logic that means “every person loves someone else.”

Question: How many of the following first-order logic statements are correct translations of "**everyone loves someone else**"?

Respond at pollev.com/zhenglian740

$$\forall p.\ (Person(p) \rightarrow \exists q.\ (Person(q) \land Loves(p, q)))$$

$$\forall p.\ (Person(p) \land \exists q.\ (Person(q) \land p \neq q \land Loves(p, q)))$$

$$\forall p.\ (Person(p) \rightarrow \exists q.\ (Person(q) \land p \neq q \land Loves(p, q)))$$

$$\exists p.\ (Person(p) \land \forall q.\ (Person(q) \land p \neq q \rightarrow Loves(q, p)))$$

Every person loves someone else

Every person loves some other person

Every person p loves some other person

Every person p loves some other person

“All As are Bs”

$$\forall x.\ (A(x) \to B(x))$$

$\forall p.\ (Person(p) \to$

p loves some other person

$)$

“All As are Bs”

$$\boldsymbol{\forall x.\ (A(x) \to B(x))}$$

$\forall p.\ (Person(p) \rightarrow$

p loves some other person

$)$

$\forall p. \; (Person(p) \rightarrow$

there is some other person that p loves

$)$

$\forall p. \ (Person(p) \rightarrow$

there is a person other than p that p loves

$)$

$\forall p.\ (Person(p) \rightarrow$

there is a person q, other than p, where p loves q

$)$

$\forall p. \ (Person(p) \to$

there is a person q, other than p, where

p loves q

$)$

$\forall p.\ (Person(p) \rightarrow$

there is a person q, other than p, where

p loves q

$)$

"Some *A*s are *B*s"

$\boldsymbol{\exists x.\ (A(x) \land B(x))}$

$\forall p.\ (Person(p) \rightarrow$

$\quad \exists q.\ (Person(q) \wedge$, *other than p, where*

p loves q

$\quad)$

$)$

"Some *A*s are *B*s"

$\boldsymbol{\exists x.\ (A(x) \wedge B(x))}$

$\forall p. \, (Person(p) \rightarrow$

$\quad \exists q. \, (Person(q) \land$, *other than p, where*

p loves q

$\quad)$

$)$

$$\forall p.\ (Person(p) \to$$
$$\quad \exists q.\ (Person(q) \land p \neq q \land$$
$$\qquad \textit{p loves q}$$
$$\quad)$$
$$)$$

$$
\begin{aligned}
&\forall p.\ (Person(p) \to \\
&\quad \exists q.\ (Person(q) \land p \neq q \land \\
&\qquad Loves(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

Using the predicates

- *Person*(p), which states that p is a person, and
- *Loves*(x, y), which states that x loves y,

write a sentence in first-order logic that means “there is a person that everyone else loves.”

There is a person that everyone else loves

There is a person p where everyone else loves p

There is a person p where everyone else loves p

“Some *A*s are *B*s”

$$\exists x.\ (A(x) \land B(x))$$

$\exists p.\ (Person(p)\ \land$

everyone else loves p

$)$

"Some *A*s are *B*s"

$\boldsymbol{\exists x.\ (A(x)\ \land\ B(x))}$

$\exists p.\ (Person(p)\ \wedge$

everyone else loves p

$)$

$\exists p.\ (Person(p)\ \wedge$

every other person q loves p

$)$

$\exists p.\ (Person(p)\ \land$

every person q, other than p, loves p

$)$

$\exists p.\ (Person(p) \land$

every person q, other than p, loves p

$)$

"All As are Bs"

$$\forall x.\ (A(x) \to B(x))$$

$\exists p. \ (Person(p) \land$

$\quad \forall q. \ (Person(q) \land p \neq q \rightarrow$

$\quad\quad$ *q loves p*

$\quad)$

$)$

"All *A*s are *B*s"

$\boldsymbol{\forall x. \ (A(x) \rightarrow B(x))}$

$$
\begin{array}{l}
\exists p.\ (Person(p)\ \land \\
\quad \forall q.\ (Person(q)\ \land\ p \neq q \rightarrow \\
\qquad \textit{q loves p} \\
\quad) \\
)
\end{array}
$$

$$
\begin{aligned}
&\exists p.\ (Person(p)\ \land \\
&\quad \forall q.\ (Person(q)\ \land\ p \neq q \to \\
&\qquad Loves(q, p) \\
&\quad) \\
&)
\end{aligned}
$$

$$
\begin{aligned}
&\forall p.\ (Person(p) \rightarrow \\
&\quad \exists q.\ (Person(q) \land \\
&\quad\quad Loves(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

$$
\begin{aligned}
&\forall p.\ (Person(p) \land \\
&\quad \exists q.\ (Person(q) \land p \neq q \land \\
&\quad\quad Loves(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

$$
\begin{aligned}
&\forall p.\ (Person(p) \rightarrow \\
&\quad \exists q.\ (Person(q) \land p \neq q \land \\
&\quad\quad Loves(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

$$
\begin{aligned}
&\exists p.\ (Person(p) \land \\
&\quad \forall q.\ (Person(q) \land p \neq q \rightarrow \\
&\quad\quad Loves(q, p) \\
&\quad) \\
&)
\end{aligned}
$$

$$\begin{array}{l}\forall p.\ (Person(p) \to \\ \quad \exists q.\ (Person(q) \land \\ \quad\quad Loves(p, q) \\ \quad) \\ )\end{array}$$

$$\begin{array}{l}\forall p.\ (Person(p) \land \\ \quad \exists q.\ (Person(q) \land p \neq q \land \\ \quad\quad Loves(p, q) \\ \quad) \\ )\end{array}$$

$$\begin{array}{l}\forall p.\ (Person(p) \to \\ \quad \exists q.\ (Person(q) \land p \neq q \land \\ \quad\quad Loves(p, q) \\ \quad) \\ )\end{array}$$

$$\begin{array}{l}\exists p.\ (Person(p) \land \\ \quad \forall q.\ (Person(q) \land p \neq q \to \\ \quad\quad Loves(q, p) \\ \quad) \\ )\end{array}$$

Everyone loves someone else

$$
\begin{aligned}
&\forall p.\ (Person(p) \rightarrow \\
&\quad \exists q.\ (Person(q) \land \\
&\quad\quad Loves(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

$$
\begin{aligned}
&\forall p.\ (Person(p) \land \\
&\quad \exists q.\ (Person(q) \land p \neq q \land \\
&\quad\quad Loves(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

$$
\begin{aligned}
&\forall p.\ (Person(p) \rightarrow \\
&\quad \exists q.\ (Person(q) \land p \neq q \land \\
&\quad\quad Loves(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

$$
\begin{aligned}
&\exists p.\ (Person(p) \land \\
&\quad \forall q.\ (Person(q) \land p \neq q \rightarrow \\
&\quad\quad Loves(q, p) \\
&\quad) \\
&)
\end{aligned}
$$

Everyone loves someone else

There is a person that everyone else loves

$$\forall p.\ (Person(p) \to$$
$$\quad \exists q.\ (Person(q) \land$$
$$\qquad Loves(p, q)$$
$$\quad)$$
$$)$$

Everyone loves someone

$$\forall p.\ (Person(p) \land$$
$$\quad \exists q.\ (Person(q) \land p \neq q \land$$
$$\qquad Loves(p, q)$$
$$\quad)$$
$$)$$

$$\forall p.\ (Person(p) \to$$
$$\quad \exists q.\ (Person(q) \land p \neq q \land$$
$$\qquad Loves(p, q)$$
$$\quad)$$
$$)$$

Everyone loves someone else

$$\exists p.\ (Person(p) \land$$
$$\quad \forall q.\ (Person(q) \land p \neq q \to$$
$$\qquad Loves(q, p)$$
$$\quad)$$
$$)$$

There is a person that everyone else loves

$$\begin{aligned}&\forall p.\ (Person(p) \to \\ &\quad \exists q.\ (Person(q) \land \\ &\qquad Loves(p, q) \\ &\quad) \\ &)\end{aligned}$$

Everyone loves someone

$$\begin{aligned}&\forall p.\ (Person(p) \land \\ &\quad \exists q.\ (Person(q) \land p \neq q \land \\ &\qquad Loves(p, q) \\ &\quad) \\ &)\end{aligned}$$

Everything is a person and and loves someone else

$$\begin{aligned}&\forall p.\ (Person(p) \to \\ &\quad \exists q.\ (Person(q) \land p \neq q \land \\ &\qquad Loves(p, q) \\ &\quad) \\ &)\end{aligned}$$

Everyone loves someone else

$$\begin{aligned}&\exists p.\ (Person(p) \land \\ &\quad \forall q.\ (Person(q) \land p \neq q \to \\ &\qquad Loves(q, p) \\ &\quad) \\ &)\end{aligned}$$

There is a person that everyone else loves

Combining Quantifiers

- Most interesting statements in first-order logic require a combination of quantifiers.
- Example: "Every person loves someone else"

For every person…	$\forall p.\ (Person(p) \to$
… there is another person …	$\exists q.\ (Person(q) \land p \neq q \land$
… they love	$Loves(p, q)$
	$)$
	$)$

Combining Quantifiers

- Most interesting statements in first-order logic require a combination of quantifiers.
- Example: “There is someone everyone else loves.”

There is a person… $\exists p.\ (Person(p)\ \land$

… that everyone else … $\forall q.\ (Person(q)\ \land\ p \neq q \rightarrow$

… loves. $Loves(q, p)$

$)$

$)$

For Comparison

For every person…

… there is another person …

… they love

$$
\begin{aligned}
&\forall p.\ (Person(p) \to \\
&\quad \exists q.\ (Person(q) \land p \neq q \land \\
&\quad\quad Loves(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

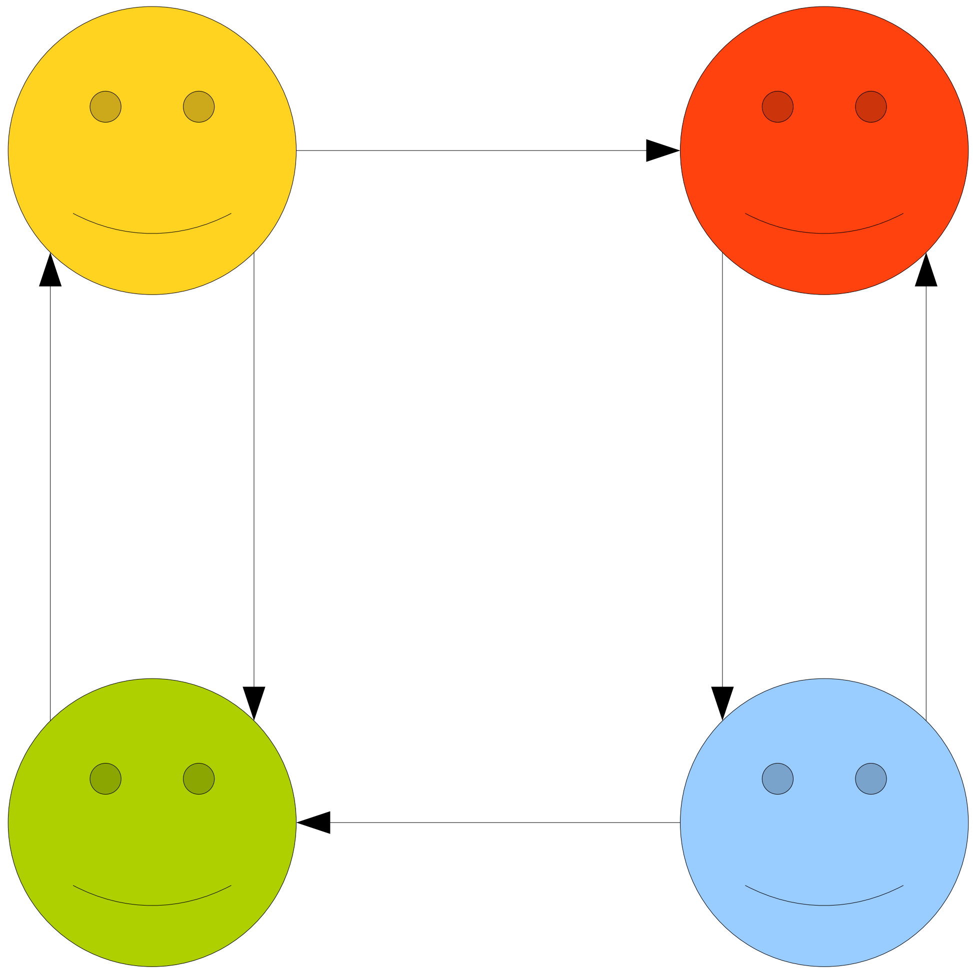
There is a person…

… that everyone else …

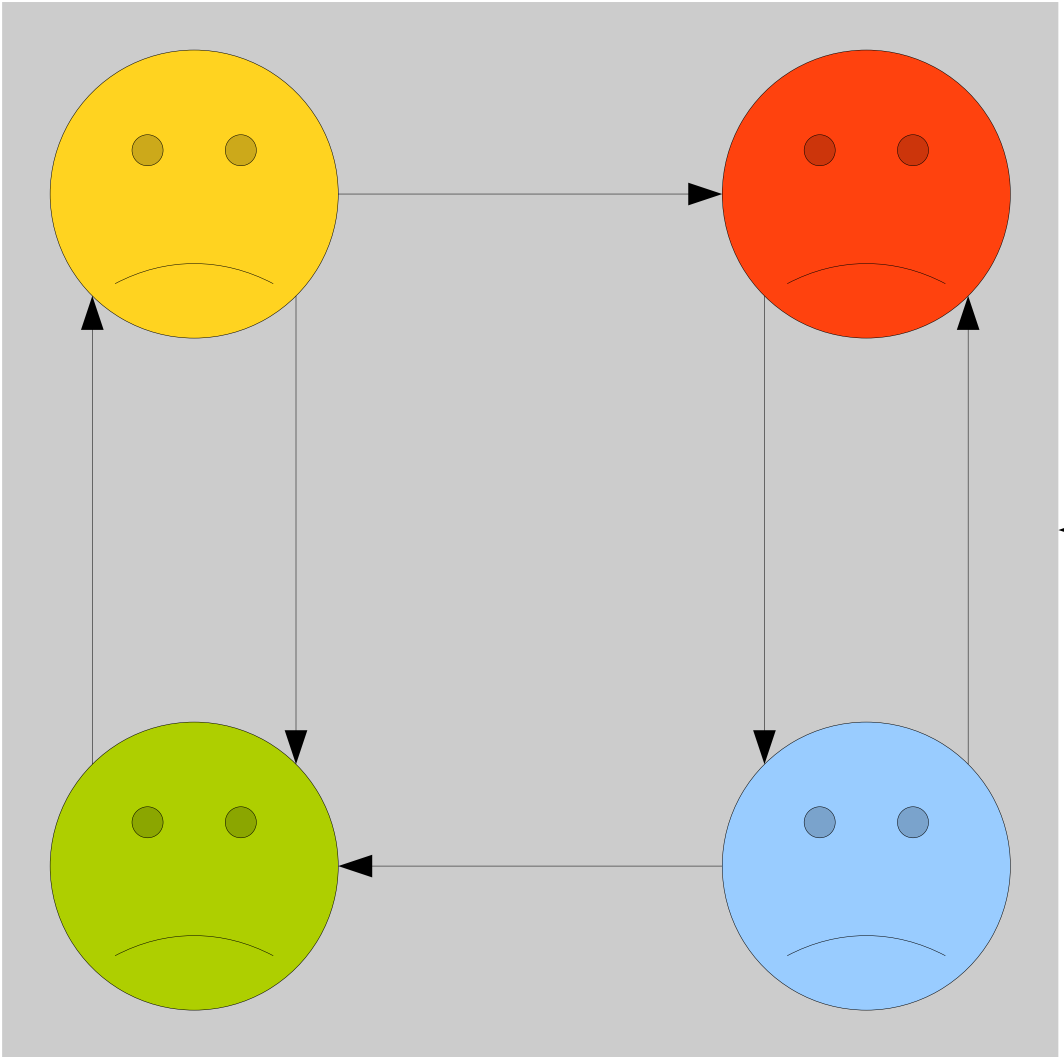
… loves.

$$
\begin{aligned}
&\exists p.\ (Person(p) \land \\
&\quad \forall q.\ (Person(q) \land p \neq q \to \\
&\quad\quad Loves(q, p) \\
&\quad) \\
&)
\end{aligned}
$$

Every Person Loves Someone Else

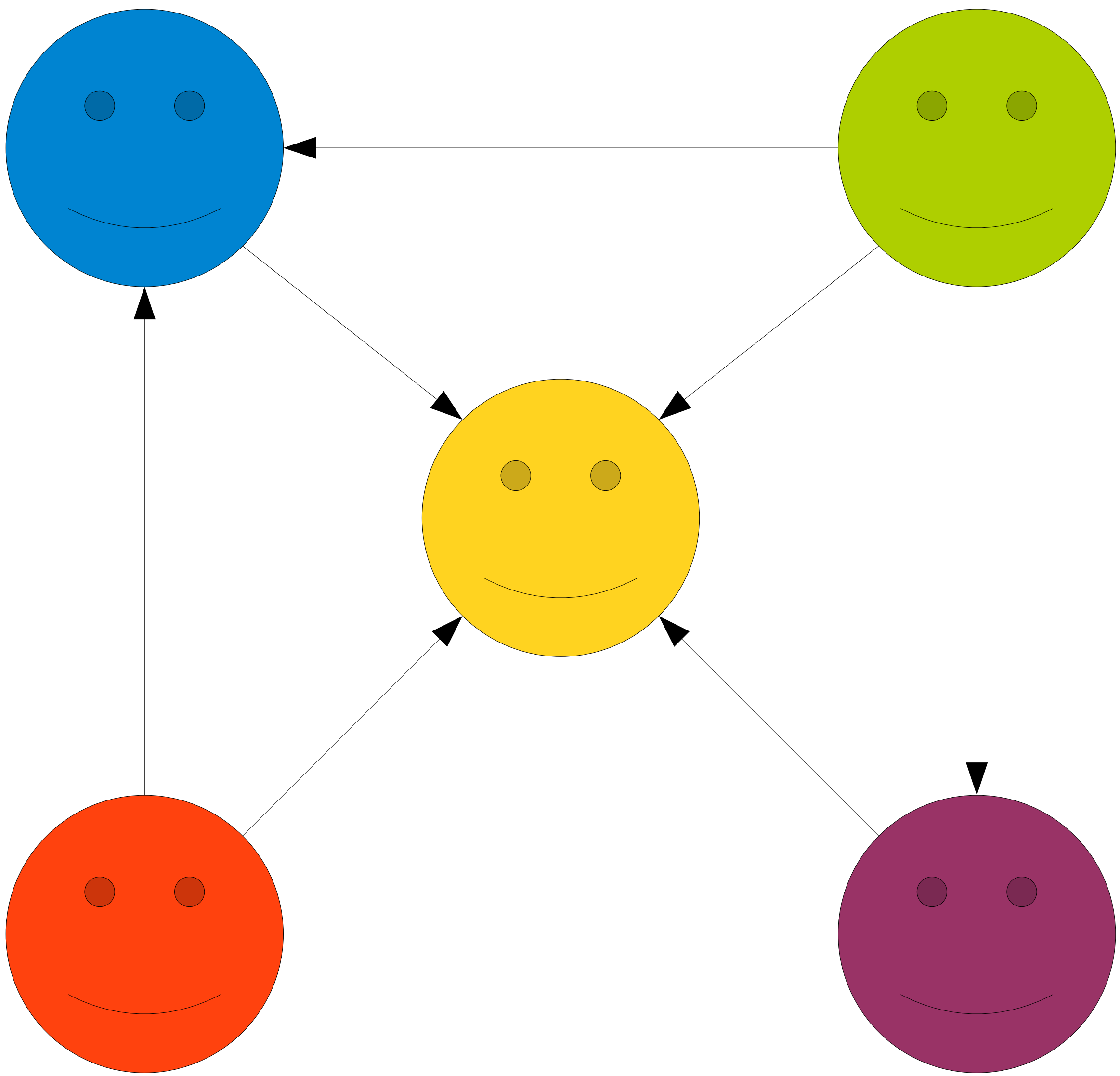


Every Person Loves Someone Else

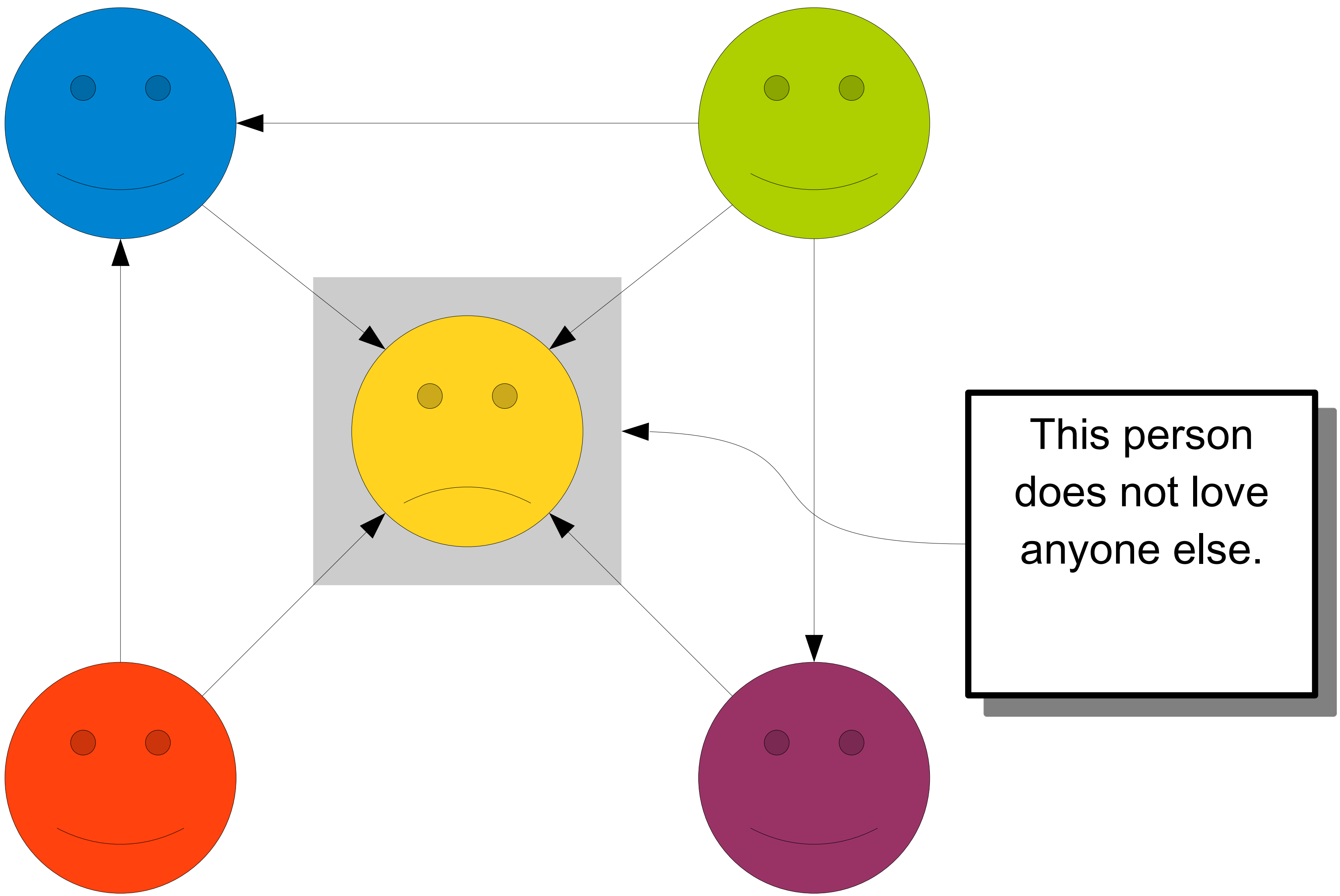


No one here is universally loved.

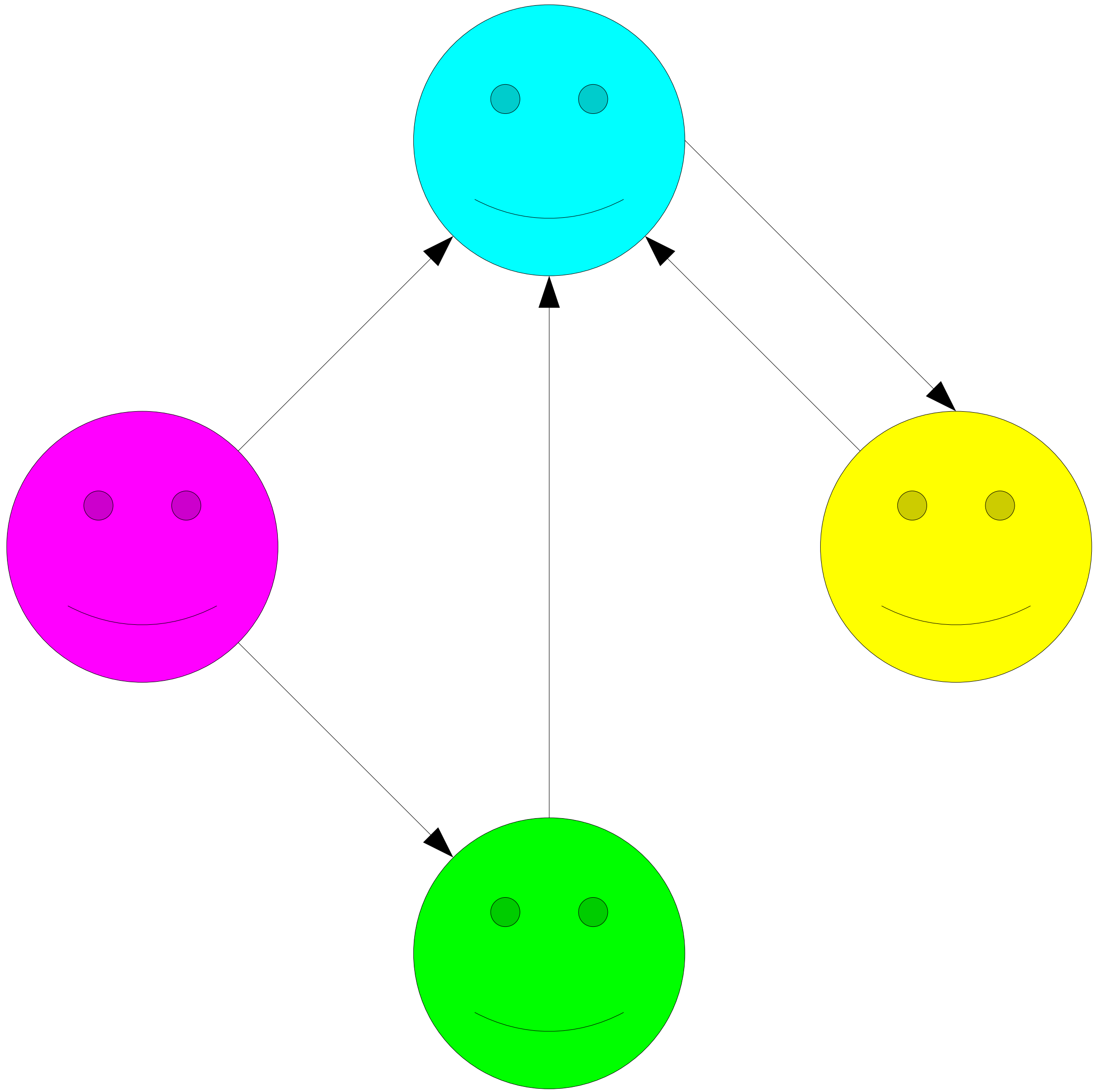
There is Someone Everyone Else Loves



There is Someone Everyone Else Loves



Every Person Loves Someone Else ***and*** There is Someone Everyone Else Loves



For every person…	$\forall p.\ (Person(p) \rightarrow$
… there is another person …	$\quad \exists q.\ (Person(q) \land p \neq q \land$
… they love	$\quad\quad Loves(p, q)$
	$\quad)$
	$)$
and	$\boldsymbol{\land}$
There is a person…	$\exists p.\ (Person(p) \land$
… that everyone else …	$\quad \forall q.\ (Person(q) \land p \neq q \rightarrow$
… loves.	$\quad\quad Loves(q, p)$
	$\quad)$
	$)$

Quantifier Ordering

- The statement

$$\forall x.\ \exists y.\ P(x, y)$$

  means "for any choice of x, there's some choice of y where $P(x, y)$ is true."
- The choice of y can be different every time and can depend on x.

Quantifier Ordering

- The statement

  $$\exists x.\ \forall y.\ P(x, y)$$

  means "there is some *x* where for any choice of *y*, we get that *P*(*x*, *y*) is true."
- Since the inner part has to work for any choice of *y*, this places a lot of constraints on what *x* can be.

Order matters when mixing existential and universal quantifiers!

Time-Out for Announcements!

Problem Set One

- Problem Set One was due today at 5:30PM.
  - Didn't submit by then? Ping us ASAP.
- Reminder: partners should make a *single* Gradescope submission and make sure to add both partners to the submission group.
- Reminder: tag your pages on Gradescope!
  - (Can do either of these things after the deadline if you forgot.)

Problem Set Two

- Problem Set Two goes out today. It's due next Friday at 5:30PM.
  - Explore first-order logic, and expand your proofwriting repertoire.
- We have some online readings for this problem set.
  - Check out the ***Guide to Logic Translations*** for more on how to convert from English to FOL.
  - Check out the ***Guide to Negations*** for information about how to negate formulas.
  - Check out the ***First-Order Translation Checklist*** for details on how to check your work.

Back to CS103!

Set Translations

Using the predicates

- $Set(S)$, which states that S is a set, and
- $x \in y$, which states that x is an element of y,

write a sentence in first-order logic that means “the empty set exists.”

Using the predicates

- $Set(S)$, which states that S is a set, and
- $x \in y$, which states that x is an element of y,

write a sentence in first-order logic that means "the empty set exists."

First-order logic doesn't have set operators or symbols "built in." If we only have the predicates given above, how might we describe this?

·***Question:*** How many of the following first-order logic statements are correct translations of “the empty set exists”?

Respond at pollev.com/zhenglian740

$$\exists S.\ (Set(S) \land \\ \quad \neg\exists x.\ x \in S \\ )$$

$$\exists S.\ (Set(S) \land \\ \quad \neg\exists x.\ x \notin S \\ )$$

$$\exists S.\ (Set(S) \land \\ \quad \neg\forall x.\ x \in S \\ )$$

$$\exists S.\ (Set(S) \land \\ \quad \neg\forall x.\ x \notin S \\ )$$

The empty set exists.

There is some set S that is empty.

$\exists S.\ (Set(S)\ \land$

S is empty.

$)$

$\exists S.\ (Set(S)\ \land$

there are no elements in S

$)$

$\exists S.\ (Set(S)\ \wedge$

$\neg$*there is an element in S*

$)$

$\exists S.\ (Set(S)\ \wedge$

$\neg$*there is an element x in S*

$)$

$$
\begin{aligned}
&\exists S.\ (Set(S)\ \wedge \\
&\quad \neg\exists x.\ x \in S \\
&)
\end{aligned}
$$

$$\exists S.\ (Set(S) \land \neg \exists x.\ x \in S)$$

$\exists S.\ (Set(S) \land \neg\exists x.\ x \in S)$

$\exists S.\ (Set(S) \land$

there are no elements in S

$)$

$$\exists S.\ (Set(S) \land \neg\exists x.\ x \in S)$$

$\exists S.\ (Set(S) \land$

every object does not belong to S

$)$

$\exists S.\ (Set(S) \land \neg\exists x.\ x \in S)$

$\exists S.\ (Set(S) \land$

every object x does not belong to S

$)$

$$\exists S.\ (Set(S) \land \neg\exists x.\ x \in S)$$

$$\begin{aligned}
&\exists S.\ (Set(S) \land \\
&\qquad \forall x.\ x \notin S \\
&)
\end{aligned}$$

$$\exists S.\ (Set(S) \land \neg\exists x.\ x \in S)$$

$$\exists S.\ (Set(S) \land \forall x.\ x \notin S)$$

$$\exists S.\ (Set(S) \land \neg \exists x.\ x \in S)$$

$$\exists S.\ (Set(S) \land \forall x.\ x \notin S)$$

Both of these translations are correct. Just like in propositional logic, there are many different equivalent ways of expressing the same statement in first-order logic.

$$\exists S.\ (Set(S) \land \neg\exists x.\ x \in S)$$

$$\exists S.\ (Set(S) \land \forall x.\ x \notin S)$$

$$\exists S.\ (Set(S) \land \neg \exists x.\ x \in S)$$

$$\exists S.\ (Set(S) \land \forall x.\ x \notin S)$$

Why can we switch which quantifier we're using here?

Mechanics: Negating Statements

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For all choices of x, $P(x)$	For some choice of x, $\neg P(x)$
$\exists x.\ P(x)$	For some choice of x, $P(x)$	For all choices of x, $\neg P(x)$
$\forall x.\ \neg P(x)$	For all choices of x, $\neg P(x)$	For some choice of x, $P(x)$
$\exists x.\ \neg P(x)$	For some choice of x, $\neg P(x)$	For all choices of x, $P(x)$

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For all choices of x, $P(x)$	**For some choice of x, $\neg P(x)$**
$\exists x.\ P(x)$	For some choice of x, $P(x)$	For all choices of x, $\neg P(x)$
$\forall x.\ \neg P(x)$	For all choices of x, $\neg P(x)$	For some choice of x, $P(x)$
$\exists x.\ \neg P(x)$	**For some choice of x, $\neg P(x)$**	For all choices of x, $P(x)$

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For all choices of x, $P(x)$	$\exists x.\ \neg P(x)$
$\exists x.\ P(x)$	For some choice of x, $P(x)$	For all choices of x, $\neg P(x)$
$\forall x.\ \neg P(x)$	For all choices of x, $\neg P(x)$	For some choice of x, $P(x)$
$\exists x.\ \neg P(x)$	**For some choice of x, $\neg P(x)$**	For all choices of x, $P(x)$

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For all choices of x, $P(x)$	$\boldsymbol{\exists x.\ \neg P(x)}$
$\exists x.\ P(x)$	For some choice of x, $P(x)$	For all choices of x, $\neg P(x)$
$\forall x.\ \neg P(x)$	For all choices of x, $\neg P(x)$	For some choice of x, $P(x)$
$\exists x.\ \neg P(x)$	For some choice of x, $\neg P(x)$	For all choices of x, $P(x)$

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For all choices of x, $P(x)$	$\exists x.\ \neg P(x)$
$\exists x.\ P(x)$	For some choice of x, $P(x)$	**For all choices of x, $\neg P(x)$**
$\forall x.\ \neg P(x)$	**For all choices of x, $\neg P(x)$**	For some choice of x, $P(x)$
$\exists x.\ \neg P(x)$	For some choice of x, $\neg P(x)$	For all choices of x, $P(x)$

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For all choices of x, $P(x)$	$\exists x.\ \neg P(x)$
$\exists x.\ P(x)$	For some choice of x, $P(x)$	$\forall x.\ \neg P(x)$
$\forall x.\ \neg P(x)$	**For all choices of x, $\neg P(x)$**	For some choice of x, $P(x)$
$\exists x.\ \neg P(x)$	For some choice of x, $\neg P(x)$	For all choices of x, $P(x)$

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For all choices of x, $P(x)$	$\boldsymbol{\exists x.\ \neg P(x)}$
$\exists x.\ P(x)$	For some choice of x, $P(x)$	$\boldsymbol{\forall x.\ \neg P(x)}$
$\forall x.\ \neg P(x)$	For all choices of x, $\neg P(x)$	For some choice of x, $P(x)$
$\exists x.\ \neg P(x)$	For some choice of x, $\neg P(x)$	For all choices of x, $P(x)$

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For all choices of x, $P(x)$	$\exists x.\ \neg P(x)$
$\exists x.\ P(x)$	**For some choice of x, $P(x)$**	$\forall x.\ \neg P(x)$
$\forall x.\ \neg P(x)$	For all choices of x, $\neg P(x)$	**For some choice of x, $P(x)$**
$\exists x.\ \neg P(x)$	For some choice of x, $\neg P(x)$	For all choices of x, $P(x)$

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For all choices of x, $P(x)$	$\exists x.\ \neg P(x)$
$\exists x.\ P(x)$	**For some choice of x, $P(x)$**	$\forall x.\ \neg P(x)$
$\forall x.\ \neg P(x)$	For all choices of x, $\neg P(x)$	$\exists x.\ P(x)$
$\exists x.\ \neg P(x)$	For some choice of x, $\neg P(x)$	For all choices of x, $P(x)$

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For all choices of x, $P(x)$	$\exists x.\ \neg P(x)$
$\exists x.\ P(x)$	For some choice of x, $P(x)$	$\forall x.\ \neg P(x)$
$\forall x.\ \neg P(x)$	For all choices of x, $\neg P(x)$	$\exists x.\ P(x)$
$\exists x.\ \neg P(x)$	For some choice of x, $\neg P(x)$	For all choices of x, $P(x)$

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	**For all choices of *x*, *P*(*x*)**	$\exists x.\ \neg P(x)$
$\exists x.\ P(x)$	For some choice of x, $P(x)$	$\forall x.\ \neg P(x)$
$\forall x.\ \neg P(x)$	For all choices of x, $\neg P(x)$	$\exists x.\ P(x)$
$\exists x.\ \neg P(x)$	For some choice of x, $\neg P(x)$	**For all choices of *x*, *P*(*x*)**

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	**For all choices of x, $P(x)$**	$\exists x.\ \neg P(x)$
$\exists x.\ P(x)$	For some choice of x, $P(x)$	$\forall x.\ \neg P(x)$
$\forall x.\ \neg P(x)$	For all choices of x, $\neg P(x)$	$\exists x.\ P(x)$
$\exists x.\ \neg P(x)$	For some choice of x, $\neg P(x)$	$\forall x.\ P(x)$

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For all choices of x, $P(x)$	$\exists x.\ \neg P(x)$
$\exists x.\ P(x)$	For some choice of x, $P(x)$	$\forall x.\ \neg P(x)$
$\forall x.\ \neg P(x)$	For all choices of x, $\neg P(x)$	$\exists x.\ P(x)$
$\exists x.\ \neg P(x)$	For some choice of x, $\neg P(x)$	$\forall x.\ P(x)$

Negating First-Order Statements

- Use the equivalences

$$\neg\forall x.\ A \quad \textit{is equivalent to} \quad \exists x.\ \neg A$$

$$\neg\exists x.\ A \quad \textit{is equivalent to} \quad \forall x.\ \neg A$$

  to negate quantifiers.
- Mechanically:
  - Push the negation across the quantifier.
  - Change the quantifier from ∀ to ∃ or vice-versa.
- Use techniques from propositional logic to negate connectives.

Taking a Negation

$$\forall x.\ \exists y.\ Loves(x, y)$$

("Everyone loves someone.")

$$\neg\forall x.\ \exists y.\ Loves(x, y)$$
$$\exists x.\ \neg\exists y.\ Loves(x, y)$$
$$\exists x.\ \forall y.\ \neg Loves(x, y)$$

("There's someone who doesn't love anyone.")

Two Useful Equivalences

- The following equivalences are useful when negating statements in first-order logic:

$$\neg(p \land q) \quad \textit{is equivalent to} \quad p \to \neg q$$

$$\neg(p \to q) \quad \textit{is equivalent to} \quad p \land \neg q$$

- These identities are useful when negating statements involving quantifiers.
  - $\land$ is used in existentially-quantified statements.
  - $\to$ is used in universally-quantified statements.
- When pushing negations across quantifiers, we *strongly recommend* using the above equivalences to keep $\to$ with $\forall$ and $\land$ with $\exists$.

Negating Quantifiers

- What is the negation of the following statement, which says "there is a cute puppy"?

$$\boldsymbol{\exists x.\ (Puppy(x) \land Cute(x))}$$

- We can obtain it as follows:

$$\neg\exists x.\ (Puppy(x) \land Cute(x))$$

$$\forall x.\ \neg(Puppy(x) \land Cute(x))$$

$$\forall x.\ (Puppy(x) \to \neg Cute(x))$$

- This says "no puppy is cute."
- Do you see why this is the negation of the original statement from both an intuitive and formal perspective?

$$\exists S.\ (Set(S) \land \forall x.\ \neg(x \in S))$$

(*"There is a set with no elements."*)

$$\neg\exists S.\ (Set(S) \land \forall x.\ \neg(x \in S))$$
$$\forall S.\ \neg(Set(S) \land \forall x.\ \neg(x \in S))$$
$$\forall S.\ (Set(S) \to \neg\forall x.\ \neg(x \in S))$$
$$\forall S.\ (Set(S) \to \exists x.\ \neg\neg(x \in S))$$
$$\forall S.\ (Set(S) \to \exists x.\ x \in S)$$

(*"Every set contains at least one element."*)

Restricted Quantifiers

Quantifying Over Sets

- The notation

$$\forall x \in S.\ P(x)$$

  means "for any element x of set S, $P(x)$ holds." (It's vacuously true if S is empty.)

- The notation

$$\exists x \in S.\ P(x)$$

  means "there is an element x of set S where $P(x)$ holds." (It's false if S is empty.)

Quantifying Over Sets

- The syntax

$$\boldsymbol{\forall x \in S.\ P(x)}$$

$$\boldsymbol{\exists x \in S.\ P(x)}$$

  is allowed for quantifying over sets.
- In CS103, feel free to use these restricted quantifiers, but please do not use variants of this syntax.
- For example, don't do things like this:

⚠ $\forall x$ with $P(x).\ Q(x)$ ⚠

⚠ $\forall y$ such that $P(y) \land Q(y).\ R(y).$ ⚠

⚠ $\exists P(x).\ Q(x)$ ⚠

Expressing Uniqueness

Using the predicate

- $WayToFindOut(w)$, which states that w is a way to find out,

write a sentence in first-order logic that means “there is only one way to find out.”

There is only one way to find out.

Something is a way to find out, and nothing else is.

Some thing w is a way to find out, and nothing else is.

Some thing w is a way to find out, and nothing besides w is a way to find out

$\exists w.\ (WayToFindOut(w)\ \wedge$

nothing besides w is way to find out

$)$

$\exists w.\ (WayToFindOut(w)\ \wedge$

anything that isn't w isn't a way to find out

$)$

$\exists w.\ (\mathit{WayToFindOut}(w)\ \land$

any thing x that isn't w isn't a way to find out

$)$

$$
\begin{array}{l}
\exists w.\ (\mathit{WayToFindOut}(w)\ \land \\
\quad \forall x.\ (x \neq w \to \textit{x isn't a way to find out}) \\
)
\end{array}
$$

$$
\begin{aligned}
&\exists w.\ (\mathit{WayToFindOut}(w) \wedge \\
&\quad \forall x.\ (x \neq w \rightarrow \neg \mathit{WayToFindOut}(x)) \\
&)
\end{aligned}
$$

$$
\begin{aligned}
&\exists w.\ (\mathit{WayToFindOut}(w)\ \land \\
&\qquad \forall x.\ (x \neq w \rightarrow \neg \mathit{WayToFindOut}(x)) \\
&)
\end{aligned}
$$

$$
\begin{aligned}
&\exists w.\ (\mathit{WayToFindOut}(w)\ \land \\
&\quad \forall x.\ (\mathit{WayToFindOut}(x) \to x = w) \\
&)
\end{aligned}
$$

$$
\begin{aligned}
&\exists w.\ (\mathit{WayToFindOut}(w)\ \wedge \\
&\quad \forall x.\ (\mathit{WayToFindOut}(x) \rightarrow x = w) \\
&)
\end{aligned}
$$

Expressing Uniqueness

- To express the idea that there is exactly one object with some property, we write that
  - there exists at least one object with that property, and that
  - there are no other objects with that property.
- You sometimes see a special "uniqueness quantifier" used to express this:

$$\exists! x.\ P(x)$$

- For the purposes of CS103, please do not use this quantifier. We want to give you more practice using the regular ∀ and ∃ quantifiers.

Next Time

- ***Functions***
  - How do we model transformations and pairings?
- ***First-Order Definitions***
  - Where does first-order logic come into all of this?
- ***Proofs with Definitions***
  - How does first-order logic interact with proofs?